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REGULAR GRAPHS AND EDGE CHROMATIC NUMBER

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For any simple graph G, Vizing's Theorem [5] implies that $\Delta(G) \leqslant \chi(G) \leqslant \Delta(G)+1$, where $\Delta(G)$ is the maximum degree of a vertex in G and $\chi(G)$ is the edge chromatic number. It is of course possible to add edges to G without changing its edge chromatic number. Any graph G is a spanning subgraph of an edge maximal graph G^* such that $\chi(G^*) = \chi(G)$. Does there always exist such a graph G^* which is $\chi(G)$-regular?

We prove that if $n \geqslant 2(k^2-k+1)$, n is even, $k \geqslant 2$ and G is a connected k-regular graph on n vertices, then G is a spanning subgraph of a $(k+1)$-regular graph G^* with $\chi(G^*) = k+1$.

For any simple graph G, Vizing's Theorem [5] implies that $\Delta(G) \leqslant \chi(G) \leqslant \Delta(G)+1$ where $\Delta(G)$ is the maximum degree of a vertex in G and $\chi(G)$ is the edge chromatic number. Here we emphasize that we are *not* using the conventional notation of $\chi'(G)$. Graphs G for which $\chi(G) = \Delta(G)$ are called class 1 and graphs G for which $\chi(G) = \Delta(G)+1$ are called class 2. The reader can do no better than to refer to [3] for an excellent introduction to this subject. Probably the major reason for its interest is Tutte's conjecture that every bridgeless cubic graph of class 2 contains a subgraph homeomorphic to the Petersen graph. The truth of this conjecture would yield a direct proof of the Four Colour Theorem. Many papers (see [3] for additional references) in recent years have been directed towards finding the so called snarks i.e. cubic graphs with cyclic edge connectivity at least 4 and girth at least 5 which are class 2. Another fruitful line of investigation (again see [3]) has been into critical graphs. A graph is *critical* if it is connected, of class 2 and if the removal of any edge lowers the chromatic index.

In this paper we take yet another viewpoint. If G is any simple graph it is of course sometimes possible to add edges to G without changing its edge chromatic number. Any graph G is a spanning subgraph of an edge maximal graph G^* such that $\chi(G^*) = \chi(G)$. Does there always exist such a graph G^* which is $\chi(G)$-regular? There are, of course, trivial cases in which this is not true. For example, if $\chi(G)$ and the number of vertices of G are both odd then G^* cannot be $\chi(G)$-regular. There are, however, large classes of graphs for which this is true. We shall content ourselves with just looking at this question for regular graphs G.

Given that a k-regular connected graph G is class 2 is it always true that it is a spanning subgraph of some $(k+1)$-regular class 1 graph G^*? For example the Petersen graph P is a 3-regular class 2 graph and it is a spanning subgraph of a 4-regular class 1 graph G^* (see Fig. 1 where the extra edges of G^* are drawn with broken lines). Incidentally it is easy (but tedious) to show that *every* 4-regular graph containing P as a spanning subgraph is class 1. We prove:

Theorem 1. *If $n \geqslant 2(k^2-k+1)$, n is even, $k \geqslant 2$ and G is a connected k-regular graph with n vertices, then G is a spanning subgraph of a $(k+1)$-regular graph G^* with $\chi(G^*)=k+1$.* □

Using Theorem 1 together with an analysis of small order cases we prove:

Theorem 2. *For $2 \leqslant k \leqslant 4$, n even and $n \geqslant 2k$, any connected k-regular graph G on n vertices is a spanning subgraph of a $(k+1)$-regular graph G^* with $\chi(G^*)=k+1$ except for $G \cong K_{3,3}$.* □

Before presenting the proofs of the theorem some notation needs to be given. Notation not specifically mentioned will follow that used in [1]. For a (simple)

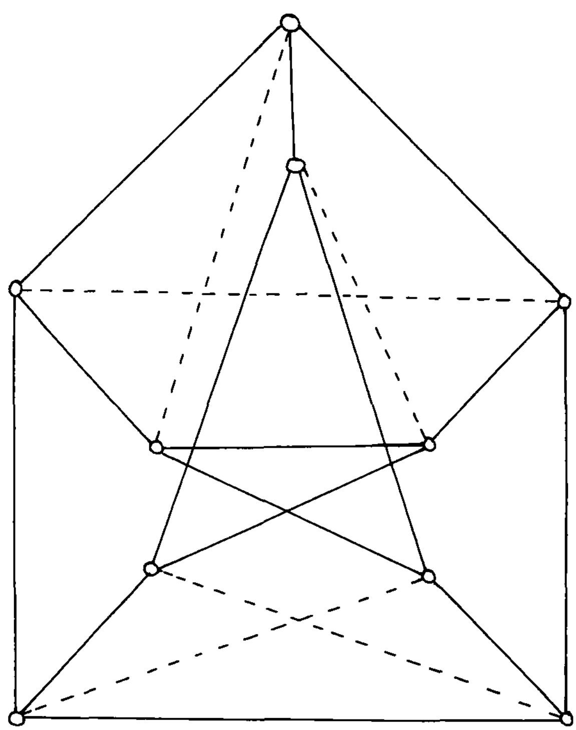

Fig. 1.

graph G, $V(G)$, $E(G)$, $\delta(G)$, and $\Delta(G)$ will denote the vertex set, edge set, minimum degree and maximum degree of G, respectively. If H and G are graphs $H \subseteq G$ will mean H is a subgraph of G and $H \leqslant G$ will mean H is a spanning subgraph of G.

In the following discussion let G be a graph on n (even) vertices with $\delta(G) \geqslant k$, $\Delta(G) \leqslant k+1$, and $\chi(G) \leqslant k+1$ $(k \geqslant 2)$. Consider a fixed $(k+1)$-edge coloring c of G, say with colors $1, 2, \ldots, k+1$. For $1 \leqslant i \leqslant k+1$, let $M_i = M_i(c)$ denote the edges which are colored i, $X_i = X_i(c) = V(G) - V(M_i)$ and $x_i = x_i(c) = \frac{1}{2}|X_i|$. Hence x_i is the number of edges needed to complete M_i to a perfect matching of G.

Since $\delta(G) \geqslant k$, clearly

(1) $\quad X_i \cap X_j = \emptyset \quad$ for $i \neq j$,

(2) $\quad \bigcup_{i=1}^{k+1} X_i \subseteq V(G)$, and

(3) $\quad \sum_{i=1}^{k+1} x_i \leqslant \frac{1}{2}n$,

with equality in both (2) and (3) if G is k-regular.

For any subset $X \subseteq V(G)$, $\langle X \rangle$ will denote the complete graph $K_{|X|}$ on vertices X, and $\langle X \rangle_i$ will denote the subgraph the i colored graph induces on X, $1 \leqslant i \leqslant k+1$. Thus if $\langle X \rangle_0$ denotes the subgraph $\bar{G}$ induces on X, then $\langle X \rangle = \bigcup_{i=0}^{k+1} \langle X \rangle_i$.

Associated with this edge coloring c of G there is a sequence $(x_1, x_2, \ldots, x_{k+1})$. With no loss of generality, we can always assume that $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_k$.

The proofs of both the theorems will generally have the following strategy. Consider a fixed edge coloring c of the k-regular graph G. Associated with the coloring c is a sequence $(x_1, x_2, \ldots, x_{k+1})$, subgraphs $\{M_i\}_{i=1}^{k+1}$ and sets $\{X_i\}_{i=1}^{k+1}$. For $2 \leqslant i \leqslant k+1$ denote by $F(X_i)$ the graph $\langle X_i \rangle - \bigcup_{j=1}^{i-1} \langle X_i \rangle_j$. In the algorithm below $F(X_i)$ is the subgraph in which we have freedom to recolor edges. First we select a perfect matching M in $\langle X_1 \rangle$ and color these edges 1. Then $M_1 \cup M$ is a perfect matching in the graph $G_1 = G \cup M \geqslant G$, and $\Delta(G_1) \leqslant k+1$. Some of the edges initially colored j for some $j > 1$ may now be colored 1. Thus, in this new edge coloring, the number x_j might be increased. Assume G_j for $j \geqslant 1$ has been chosen with edge coloring c', having sequence $(x'_1, x'_2, \ldots, x'_{k+1})$, subgraphs $\{M'_i\}_{i=1}^{k+1}$ and sets $\{X'_i\}_{i=1}^{k+1}$. We are assuming that M'_i is a perfect matching of G_j for $1 \leqslant i \leqslant j$ and that $\Delta(G_j) \leqslant k+1$. Select a perfect matching M' in $F(X'_{j+1})$ and color these edges $j+1$. Then $M'_{j+1} \cup M'$ is a perfect matching in the graph $G_{j+1} = G_j \cup M' \geqslant G_j$ and $\Delta(G_{j+1}) \leqslant k+1$. The perfect matchings in colors $1, 2, \ldots, j$ are left unchanged. If it is possible to choose a perfect matching in $F(X'_{j+1})$ for each $j \leqslant k$, this will then yield a graph $G_{k+1} \geqslant G$ which is $(k+1)$-regular and $(k+1)$-edge colorable as desired. We will say that the set X_{j+1} is *open* if $F(X_{j+1}) \geqslant x_{j+1}K_2$. Thus we are looking for colorings which give a sequence $(x_1, x_2, \ldots, x_{k+1})$ such that each X_i $(2 \leqslant i \leqslant k+1)$ is open.

We will now state and prove some lemmas that will give conditions that will ensure that these matchings can be chosen.

The following well known result of Tutte [4] will be used frequently, and so we state it here. It says that a graph G has a perfect matching if and only if for each subset $S \subseteq V(G)$, the number of components of $G-S$ with an odd number of vertices is less than or equal to $|S|$. This implies, that a graph G with an even number of vertices which does not have a perfect matching, must have a separating set S such that $G-S$ has at least $|S|+2$ odd components.

In the following two lemmas, let G be a connected graph with $\delta(G) \geq k$ and $\Delta(G) \leq k+1$. Assume that G has a $(k+1)$-edge coloring c with associated graphs $\{M_i\}_{i=1}^{k+1}$, sets $\{X_i\}_{i=1}^{k+1}$ and sequence $(x_1, x_2, \ldots, x_{k+1})$.

Lemma 3. *Let i be fixed for some i, $2 \leq i \leq k+1$.*

(i) *If $x_i \geq i$, then X_i is open.*

(ii) *If $x_i = i-1$, then one of the following holds:*

(a) *X_i is open,*

(b) *$F(X_i) \cong K_{x_i} \cup K_{x_i}$ and x_i is odd,*

(c) *$F(X_i) \leq K_{x_i-1} + \bar{K}_{x_i+1}$ and x_i is odd.* ∙

Proof. Let $H = F(X_i)$. It follows from the definition of H that $\delta(H) \geq 2x_i - i$. If $x_i \geq i$, then $\delta(H) \geq x_i$ and therefore by Dirac's Theorem [2], H contains a Hamiltonian cycle and $H \geq x_i K_2$.

Assume $x_i = i-1$ and that X_i is not open. Note that $\delta(H) \geq x_i - 1$. If H is disconnected, then Tutte's Theorem implies that x_i is odd and $H \cong K_{x_i} \cup K_{x_i}$ which gives (b).

If H is connected, then there is a separating set S with $1 \leq s = |S| \leq x_i - 1$ such that $H-S$ contains at least $s+2$ odd components. Each odd component must contain at least $x_i - s$ vertices since $\delta(H-S) \geq x_i - s - 1$, thus

$$(s+2)(x_i - s) + s \leq 2x_i.$$

This implies $s \geq x_i - 1$. Combining this with a previous inequality gives that $s = x_i - 1$, and that there are precisely $s+2 = x_i + 1$ odd components each with just one vertex. Hence

$$H \leq K_{x_i-1} + \bar{K}_{x_i+1}.$$

Thus

$$\bar{H} \supseteq K_{x_i+1} = K_i \quad \text{and} \quad \Delta(\bar{H}) \leq i-1.$$

The copy of K_i in $\bar{H}$ must have a perfect matching in each color j, $1 \leq j \leq i-1$, and so $i = x_i + 1$ is even. This verifies (c).

Remark. Lemma 3 implies that if $x_i \geq i$ or if x_i is even and $x_i = i-1$, then X_i is open. Also if G is k-regular, $x_{k+1} = k$ and G has more that $2k$ vertices, then X_{k+1} is open. If this were not true, then $\bar{H} \cong K_{k,k}$ or $\bar{H} \supseteq K_{k+1}$. Since $G \supseteq \bar{H}$, $\Delta(G) = k$

and G is connected, this would imply that $G = K_{k,k}$ or $G = K_{k+1}$, a contradiction. We will say that X_i is *strongly open* if either:

(i) $x_i \geqslant i$ $(2 \leqslant i \leqslant k+1)$,

(ii) $x_i = i-1$ and i is odd $(2 \leqslant i \leqslant k+1)$,

(iii) $i = k+1$, $x_{k+1} = k$ and G is k-regular with $n \geqslant 2k+1$, hold. By our remark if X_i is strongly open, then it is open.

Lemma 4. *If G has a $(k+1)$-edge coloring with sequence $(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_{k+1})$ and $x_i < x_j$, then G has a $(k+1)$-edge coloring with sequence $(x_1, \ldots, x_i + 1, \ldots, x_j - 1, \ldots, x_{k+1})$.*

Proof. Let $H_{ij} = M_i \cup M_j$, which is a disjoint union of paths and even length cycles. Since $x_i < x_j$, there must be at least one path which starts and ends with an edge of color i. Interchanging the colors on this path gives the desired result. □

We say that a sequence $(x_1, \ldots, x_{k+1})$ is *level* if $|x_i - x_j| \leqslant 1$ for all i and j. Lemma 4 implies that G has a $(k+1)$-edge coloring which gives a level sequence. Lemma 4 is well known. Amongst others (see [3, p. 132]), Berge, Fulkerson and De Werra have discussed level (equitable) colorings.

Proof of Theorem 1. If $k = 2$ the result is clear. In this case the graph G is just a cycle, which is a spanning subgraph of a 3-regular graph which can be 1-factored. We assume $k \geqslant 3$.

First consider the case when G contains a perfect matching. Thus Lemma 4, together with the condition on n, gives that G has a $(k+1)$-edge coloring c which produces a sequence $(0, x_2, \ldots, x_{k+1})$ with $x_i \geqslant k-1$ $(2 \leqslant i \leqslant k)$ and $x_{k+1} \geqslant k$, (where $(x_2, \ldots, x_{k+1})$ is a level sequence). All of the sets X_j are strongly open $(2 \leqslant j \leqslant k+1)$ except possibly for X_k. If X_k is open, the theorem follows in this case. Thus we can assume by Lemma 3 that $x_k = k-1$, x_k is odd (so that $k \geqslant 4$), and there is an edge $e \in \langle X_k \rangle_2$. If we change the color of this edge e to k, the value of x_k is decreased by 1. This implies that G has a $(k+1)$-edge coloring c' which produces the sequence $(0, x'_2, \ldots, x'_{k+1})$ with $x'_2 \geqslant k-2$, $x'_j \geqslant k-1$ for $3 \leqslant j \leqslant k-1$ and $x'_k, x'_{k+1} \geqslant k$. Thus all of the sets X'_j $(j \geqslant 2)$ are strongly open and the theorem follows in this case.

We assume that G does not have a perfect matching, and consider the case where G has a $(k+1)$-edge coloring giving a sequence $(x_1, \ldots, x_{k+1})$ with $x_{k+1} = k$. Using Lemma 4, together with the condition on n, we can assume that $x_i \geqslant k-2$ for $1 \leqslant i \leqslant k-1$ and $x_k \geqslant k-1$. If $k = 3$, all of the sets X_i $(1 \leqslant i \leqslant k+1)$ are open. In fact each X_i is strongly open except possibly for X_2. Now suppose $k = 3$ and X_2 is not open. Then $x_2 = 1$ and $\langle X_2 \rangle_1$ has no edges. Moreover $M \cup \{e\}$, when $e \in \langle X_2 \rangle_1$, is a perfect matching. Thus we can assume $k \geqslant 4$. All of the X_i are strongly open except for possibly one of X_{k-1} or X_k. If both are open, the result follows. If X_{k-1} is not open, then $x_{k-1} = k-2$, k is odd, and there is an edge

$e \in \langle X_{k-1}\rangle_1$. If X_k is not open, then, from Lemma 3(ii), $x_k = k-1$, k is even, and there are two independent edges $f, g \in \langle X_k\rangle_1$. Clearly both X_{k-1} and X_k cannot fail to be open. By changing the color of edge e to $k-1$ or by changing the color of edges f and g to k (depending on whether X_{k-1} or X_k fails to be open), we ensure that G has a $(k+1)$-edge coloring which gives a sequence $(x'_1, \ldots, x'_k)$ such that each of the X'_1 $(1 \leq i \leq k+1)$ is strongly open.

Finally consider a $(k+1)$-edge coloring of G that gives a level sequence $(x_1, \ldots, x_{k+1})$. If $x_{k+1} \geq k+1$, then $x_i \geq k$ for all i. Thus in this case each X_i is strongly open and the results follows. We have already considered the case when $x_{k+1} = k$. The only case which remains is when $x_{k+1} = k-1$. Because of the condition on n we have $x_i = k-2$ $(1 \leq i \leq k-2)$ and $x_{k-1} = x_k = x_{k+1} = k-1$. If there is an edge $e \in \langle X_i\rangle_j$ for $i \neq j$ and $i, j \in \{k-1, k, k+1\}$, then changing the color of e from j to i ensures that there is a $(k+1)$-edge coloring of G which has a sequence $(x'_1, x'_2, \ldots, x'_{k+1})$ with $x'_{k+1} = k$. Hence we can assume that $\langle X_i\rangle_j$ is the empty graph for $i \neq j$ and $i, j \in \{k-1, k, k+1\}$. This implies that $\delta(\langle X_j\rangle_0) \geq k-1 = x_j$ for $j = k-1, k, k+1$, and so $\langle X_j\rangle_0$ has a perfect matching (in fact a Hamiltonian cycle). Therefore each of the X_i are open and the theorem follows. □

Proof of Theorem 2. For $k=2$ the result is obvious from Theorem 1. We will give the details for the case $k=3$. The proof for $k=4$ is very similar but more tedious, thus we omit the proof of this case. So now we will assume that G is a 3-regular graph on n vertices. For $n \geq 14$, Theorem 1 implies Theorem 2. So we may assume $6 \leq n \leq 12$.

If G is class 1 and $\bar{G}$ contains a 1-factor then the theorem is proved. However if G has 6 or 8 vertices [3, p. 41], then G is class 1. Since $\bar{G}$ is a regular graph of degree $n-4$ $\bar{G}$ contains a perfect matching except when $n=6$ and $G \cong K_{3,3}$. This is the exceptional graph in the statement of the theorem.

Now suppose $n=10$. Since $\bar{G}$ is a regular graph of degree 6 it has a perfect matching. Again [3, p. 103], if G is a bridgeless graph and G is not the Petersen graph, then G is class 1 and the theorem is proved. The Petersen graph is class 2 but it is easy to check (see Fig. 1) the truth of the theorem for the Petersen graph. Again it is easy to check the theorem for the only connected cubic graph on 10 vertices with a bridge.

So now suppose $n=12$. If G has a perfect matching, then by Lemma 4 there is an edge coloring with sequence $(0, 2, 2, 2)$ and with associated sets M_i $(i = 1, 2, 3, 4)$ and X_j $(j = 1, 2, 3, 4)$. If $\langle X_i\rangle_j$ has no edges for some $i \neq j$ $(2 \leq i, j \leq 4)$, then it is easily checked (by appropriately ordering X_2, X_3 and X_4) that each of the X_i are open $(i = 2, 3, 4)$ and we are done. If $\langle X_i\rangle_j \cup \langle X_i\rangle_k \supseteq 2K_2$ for distinct $i, j, k \geq 2$, then G has a perfect matching distinct from M_1. Hence G is class 1 and, by Dirac's condition, $\bar{G}$ has a perfect matching. Again the theorem follows.

Finally we may assume that $\langle X_4\rangle_j$ has precisely one edge e_j $(j = 2, 3)$ and that e_2 and e_3 are incident. With no loss of generality we can assume that the edge f in $\langle X_4\rangle$ which is independent from e_2 is in $\langle X_4\rangle_0$. If we color the edge e_2 with color 4 instead of color 2 we obtain an edge coloring with a sequence (0, 1, 2, 3) such that each of the sets is open.

We can assume that G has no perfect matching, and that, using Lemma 4, there is an edge coloring with sequence (1, 1, 2, 2). If there is an edge $e \in \langle X_4 \rangle_3$, then coloring e with 4 gives an edge coloring c' with sequence (1, 1, 1, 3). Since G has no perfect matching, each of the sets associated with the coloring c' would have to be open, and the result would follow. If $\langle X_4 \rangle_3$ has no edges, then each of the sets X_i $(1 \leqslant i \leqslant 4)$ are open. This completes the proof. □

We suspect that Theorem 1 is weak and that the following might be true:

Conjecture. For $k \geqslant 2$, n even and $n \geqslant 2k$, any connected k-regular graph G on n vertices is a spanning subgraph of a $(k+1)$-regular graph G^* with $\chi(G^*) = k+1$, except when $G \cong K_{k,k}$ and k is odd. □

We have already observed that the Petersen graph P is a class 2 graph with the property that if G^* is any 4-regular graph such that $P \leqslant G^*$, then G^* is class 1.

Question 1. Do there exist k-regular, $(k-1)$-edge-connected class 2 graphs which are spanning subgraphs of $(k+1)$-regular class 2 graphs?

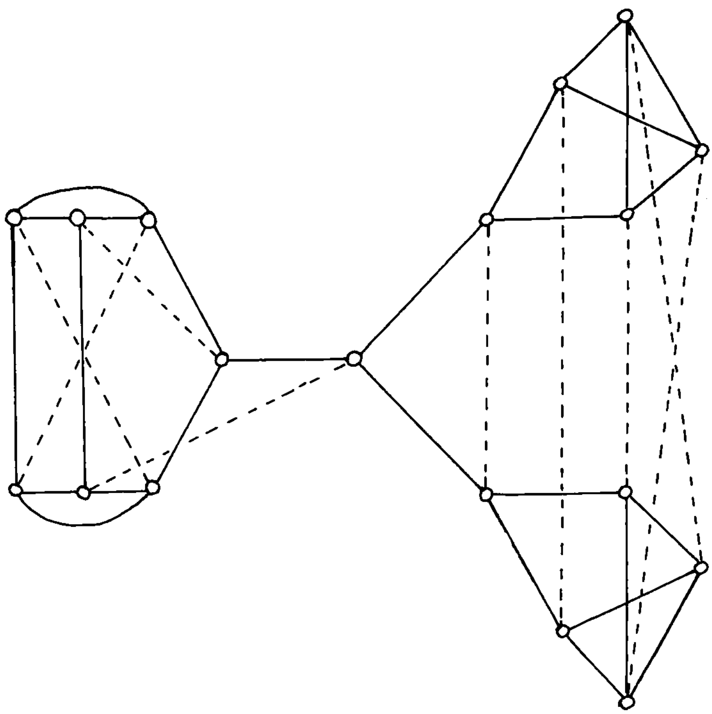

Fig. 2.

Remark. The answer is trivially "yes" if the edge-connectivity condition is dropped. If G is k-regular and has a cut vertex, then G is class 2 so that provided G^* is chosen so that G^* has a cut vertex we are done. (See Fig. 2 where the extra edges in G^* are drawn with broken lines.)

Question 2. Do there exist k-regular, $(k-1)$-edge-connected, vertex transitive, class 2 graphs which are spanning subgraphs of $(k+1)$-regular class 2 graphs?

Remark. One of the referees pointed out that the answer to Question 1 is in the affirmative.

Consider any snark S with an even number of edges. Then $L(S)$, the line graph of S, is class 2, by results of Seymour and Kotzig [3, p. 41] and $L(S)$ is 4-regular and of even order. Further $L(S)$ has a 1-factor F. So $L(S)-F$, which must be class 2, is the required cubic graph.

Acknowledgment

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